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The criterion for the attainment of maximum load for a cracked elastic-softening solid

E. SMITH
Manchester University, UMIST Materials Science Centre, Grosvenor Street, Manchester, M1 7HS, UK

This paper focuses on the criterion for the attainment of maximum load for a cracked brittle solid where the material is of the elastic-softening variety. Theoretical analysis of a simple model clearly shows that, as the initial crack size decreases, there is an increasing potential for the maximum load to be attained prior to the attainment of a fully developed softening zone.

1. Introduction

There are many brittle materials where crack extension proceeds by a mechanism in which the crack faces are bridged by unfractured ligaments which restrict the crack opening and thereby toughen the material; examples include concretes, fibre cements and non-phase-transforming ceramics. The material's fracture resistance, as manifested by the stress intensity at the propagating crack tip, increases with crack extension thereby giving a time-independent R curve behaviour (where R is the softening-zone size); this increase is due to the restraining effect of the unbroken ligaments that remain behind the tip of the propagating crack. There have been several recent theoretical studies of this type of behaviour (for example, [1–5]). A common procedure is to consider the concept that the average behaviour of the unbroken ligaments can be represented by a decreasing stress, p, versus increasing crack opening, v, relationship. Such a material can therefore be regarded as being elastic softening, and can be modelled by a softening zone, coplanar with the crack tip and extending to the initial crack-tip position. The restraining stresses reflect the softening characteristics of the material with each material being characterized by its own particular $p(v)$ law. The maximum value of p occurs at the crack tip which is the leading edge of the softening zone. When $p = 0$ at the trailing edge, i.e. the initial crack-tip position, and the crack opening v is then equal to a critical value δ_c, the softening zone is said to be fully developed. It is recognized that the maximum load state can occur prior to the attainment of a fully developed softening zone. However, very few considerations have focused on this particular point, which quantifies the factors that govern the extent to which the maximum load is attained prior to the full development of a softening zone. This is clearly an important problem since it is central to the use of laboratory test data in predicting the behaviour of cracked solids under service loading conditions.

It is against this background that this paper addresses the issue for the highly idealized two-dimensional plane-strain Mode I model where there is an initial crack of size $2a_0$ in a solid which is subjected to a uniform applied tensile stress; this model approximately describes the behaviour of a very large solid containing an edge crack of initial depth a_0. It is assumed that, upon loading the solid, the crack tip remains stationary until a critical value K_{IC} of the stress intensity is attained at the crack tip when the crack begins to extend. As the applied loading increases, the crack continues to extend, leaving behind unfractured ligaments which bridge the crack faces, their restraining effect being represented by a constant restraining stress, p_c, between the crack faces. This restraining stress is presumed to be operative until the opening between the crack faces becomes too large for the ligaments to remain unfractured; the restraint is assumed to fall from p_c to zero when the opening v attains a critical value δ_c. This description of the propagation of a crack tip can be viewed as simulating, in a highly idealized manner, the behaviour of an elastic-softening material for which the p–v softening law is such that there is a rapid fall-off from a high stress value to a much lower value p_c at a small displacement, followed by a long tail in the softening law; concrete exhibits these characteristics. For the particular case where $K_{IC} = 0$, when the model becomes equivalent to the Dugdale–Bilby–Cottrell–Swinden model [6, 7] of plastic relaxation at a crack tip, the maximum-stress state is always associated with the attainment of a fully developed ligament zone, irrespective of the initial crack size. The present paper's analysis addresses the more general $K_{IC} \neq 0$ situation, and demonstrates the extent to which the maximum-stress state can occur prior to the attainment of a fully developed softening zone, as the initial crack size decreases.

2. Theoretical analysis

Fig. 1 shows a model of an isolated two-dimensional crack of length $2a$ (initial length $2a_0$) in an infinite solid subjected to a uniform applied tensile stress, σ,

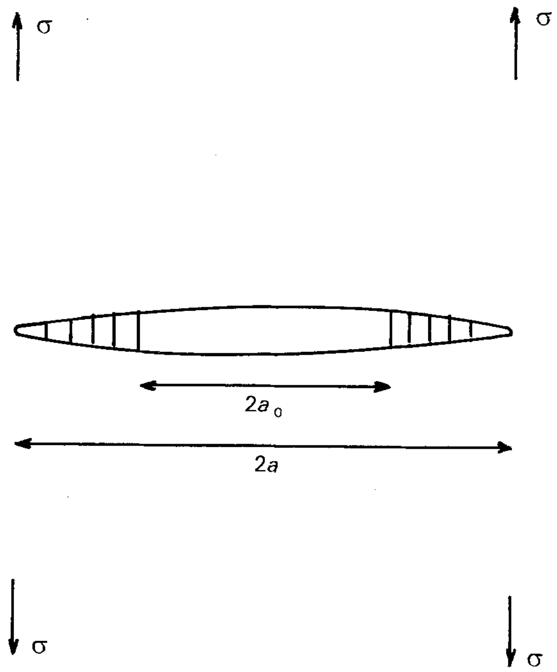

Figure 1 A model of a crack in a uniformly stressed infinite solid; there is a partially developed softening zone at each crack tip.

normal to the crack plane. There is a partially developed softening zone at each crack tip, the tensile stress within each zone being p_c; the opening displacement at the original crack tips, i.e. the trailing edges of the softening zones, is δ. By using standard results [8] for a line force applied to the faces of a crack, it is easily shown that the opening displacement, δ, is given by the expression

$$\delta = \frac{8p_c a_0}{\pi E_0}\ln\left(\frac{a}{a_0}\right) + \frac{4K_{IC}(a^2 - a_0^2)^{1/2}}{E_0(\pi a)^{1/2}} \quad (1)$$

where $E_0 = E/(1 - \nu^2)$, E is the Young's modulus of the bulk material and ν is Poisson's ratio; K_{IC} is the stress intensity at the crack tip. Furthermore, the crack-tip stress intensity, $K_A = \sigma(\pi a)^{1/2}$, due to the applied loadings is given by the expression

$$K_A = \sigma(\pi a)^{1/2} = K_{IC} + 2p_c\left(\frac{a}{\pi}\right)^{1/2} \times \cos^{-1}\left(\frac{a_0}{a}\right) \quad (2)$$

For the extreme case where the initial crack is very long, i.e. $a_0 \to \infty$, Equations 1 and 2 give (with $\delta = \delta_c$) the fully-developed-softening-zone size $R_{m\infty}$ as

$$R_{m\infty} = \frac{\pi}{8p_c^2}[(K_{IC}^2 + E_0 p_c \delta_c)^{1/2} - K_{IC}]^2 \quad (3)$$

while the value of K_A, i.e. $K_{m\infty}$, associated with the attainment of a fully developed softening zone is given by the expression

$$K_{m\infty} = (K_{IC}^2 + E_0 p_c \delta_c)^{1/2} \quad (4)$$

a result that can also be obtained directly by application of the J-integral approach [9]. For this particular case, i.e. $a_0 \to \infty$, the applied stress increases until the softening zone is fully developed when unstable crack extension occurs.

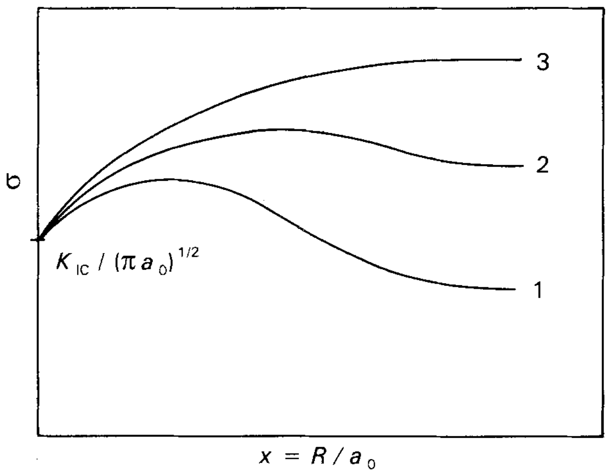

Figure 2 The schematic variation of σ with $x = R/a_0$; there are three possible types of variation.

To examine the behaviour of a solid containing a finite-size initial crack, let $x = R/a_0$ where R is the softening-zone size, when Equation 2 becomes

$$\sigma = \frac{K_{IC}}{(\pi a_0)^{1/2}(1 + x)^{1/2}} + \frac{2p_c}{\pi}\cos^{-1}\frac{1}{(1 + x)} \quad (5)$$

with three possible types of σ–x variation being shown schematically in Fig. 2. Equation 5 shows that $d\sigma/dx = 0$ and σ is a maximum when

$$\lambda = \frac{8p_c^2 a_0}{\pi K_{IC}^2} = \frac{x(2 + x)}{2(1 + x)} \quad (6)$$

which shows that σ always goes through a maximum irrespective of the value of λ unless $\lambda \to \infty$, i.e. the initial crack size is very large, when we have the Type 3 variation shown in Fig. 2. With $1 + x = \varphi$, Equation 6 shows that the maximum in σ occurs when

$$\varphi = 1 + x = \lambda + (\lambda^2 + 1)^{1/2} \quad (7)$$

Moreover, Equations 1, 6 and 7 show that, when σ attains its maximum value, the displacement, δ, at the initial crack tip is given by the expression

$$\delta = \frac{8p_c a_0}{\pi E_0}\{2 + \ln[\lambda + (\lambda^2 + 1)^{1/2}]\} \quad (8)$$

It therefore follows that the maximum in σ is reached before the softening zone is fully developed ($\delta = \delta_c$), if (see Equation 8)

$$2 + \ln\{\lambda + (\lambda^2 + 1)^{1/2}\} < \frac{\pi E_0 \delta_c}{8p_c a_0} \quad (9)$$

remembering that $\lambda = 8p_c^2 a_0/\pi K_{IC}^2$. With the aid of Equation 4, this inequality can be written in the form

$$[1 + \lambda\{2 + \ln[\lambda + (\lambda^2 + 1)^{1/2}]\}]^{1/2} < \frac{K_{m\infty}}{K_{IC}} \quad (10)$$

where the parameters $K_{m\infty}$ and K_{IC} are physically identifiable. This inequality is expressed graphically in Fig. 3, which clearly shows that the attainment of the maximum-stress state and unstable failure prior to the attainment of a fully developed softening zone are favoured by a small initial crack size and a high $K_{m\infty}/K_{IC}$ ratio.

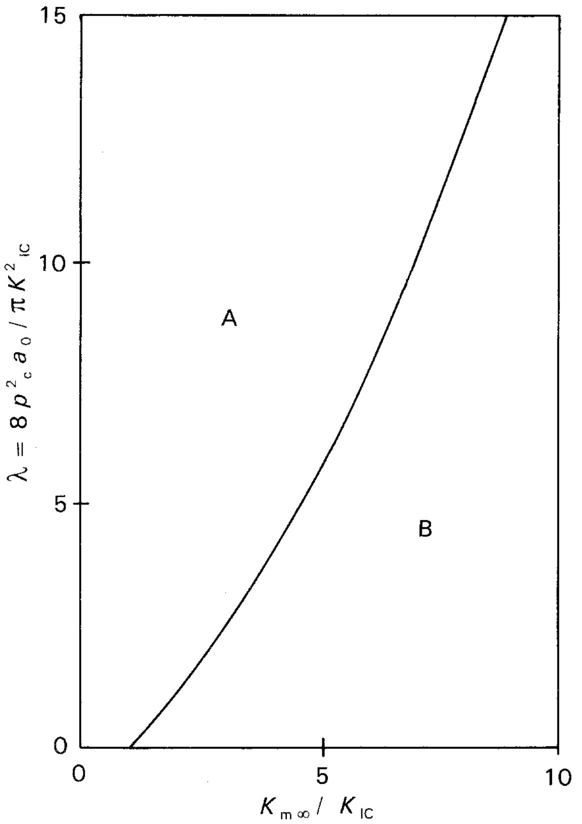

Figure 3 Maximum stress (A) upon the attainment of a fully developed softening zone, and (B) before the attainment of a fully developed softening zone.

When Inequality 10 is satisfied, Equation 5 gives the maximum stress σ_m sustainable by the solid when it contains a crack with initial size a_0, i.e.

$$\sigma_m = \frac{K_{IC}}{(\pi a_0)^{1/2}\varphi^{1/2}} + \frac{2p_c}{\pi}\cos^{-1}\varphi^{-1} \quad (11)$$

with φ being given in terms of λ by Equation 7, remembering that the parameter λ is related to a_0 (see Equation 6). If Inequality 10 is not satisfied, the maximum sustainable stress σ_m is associated with the attainment of a fully developed softening zone; in this case Equation 1 with $\delta = \delta_c$ together with Equations 2 and 4 gives

$$\sigma_m = \frac{K_{IC}(\cos\theta)^{1/2}}{(\pi a_0)^{1/2}} + \frac{2p_c\theta}{\pi} \quad (12)$$

where $\theta = \cos^{-1}(a_0/a)$ is given by the relation

$$\left[1 + \lambda\left\{\ln\sec\theta + \frac{2^{1/2}\sin\theta}{\lambda^{1/2}(\cos\theta)^{1/2}}\right\}\right]^{1/2} = \frac{K_{m\infty}}{K_{IC}} \quad (13)$$

Equations 11 and 12 relate the maximum sustainable stress to the initial crack size and they can be expressed in terms of the applied stress intensity $K_* \equiv \sigma_m(\pi a_0)^{1/2}$. Thus within the regime where the attainment of maximum stress occurs prior to the full development of a softening zone, i.e. Inequality 10 is satisfied, K_* is given by the expression

$$\frac{K_*}{K_{IC}} = \frac{1}{\varphi^{1/2}} + \left(\frac{\lambda}{2}\right)^{1/2}\cos^{-1}\left(\frac{1}{\varphi}\right) \quad (14)$$

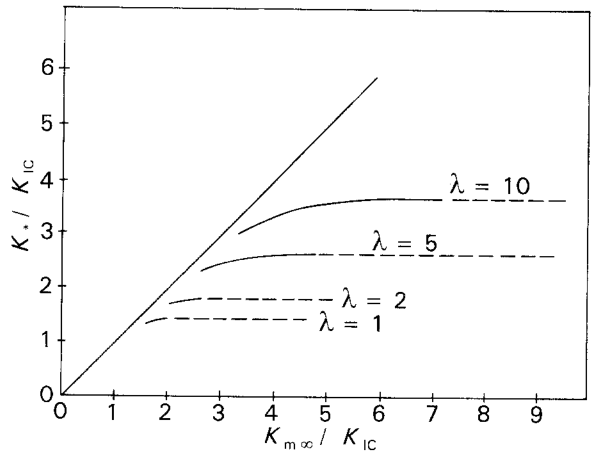

Figure 4 The effective stress intensity K_* associated with maximum load for a solid containing a flaw with initial size $2a_0$; $\lambda = 8p_c^2 a_0/\pi K_{IC}^2$.

with φ being related to λ by Equation 7. On the other hand, when the maximum stress is associated with the attainment of a fully developed softening zone, i.e. Inequality 10 is not satisfied, K_* is given by the expression

$$\frac{K_*}{K_{IC}} = (\cos\theta)^{1/2} + \left(\frac{\lambda}{2}\right)^{1/2}\theta \quad (15)$$

Results obtained from Equations 14 and 15 are given in Fig. 4 which shows the effect of initial crack size $2a_0$ on the maximum-load criterion when it is expressed in terms of an effective-stress-intensity factor K_*; K_*/K_{IC} is plotted against $K_{m\infty}/K_{IC}$ for various crack sizes, i.e. for various λ values. The full curves in Fig. 4 give the K_* values relevant to the situation where maximum load is associated with the attainment of a fully developed softening zone; these curves for the various λ values attain a maximum at the end points of the full curves. The curves then decrease and eventually level out at a lower K_*/K_{IC} value as $K_{m\infty}/K_{IC} \rightarrow \infty$; the decreasing regions (not shown in Fig. 4) correspond to maximum load being attained prior to the full development of a softening zone. The levelling out values of K_*/K_{IC} for $\lambda = 1$, 2, 5 and 10 are, respectively, 1.11, 1.57, 2.48 and 3.51, whereas the maximum values are 1.45, 1.82, 2.64 and 3.62 (these latter values are the end points of the full curves in Fig. 4).

Probably the clearest way of quantifying the attainment of the maximum-stress state prior to the full development of a softening zone is to express the attainment of the stress maximum in terms of the J-integral. Now J is given by the expression

$$J = \frac{K_{IC}^2}{E_0} + p_c\delta \quad (16)$$

with δ being given by Equation 8. Thus with the specific fracture energy G_F for the softening law being given by the relation

$$G_F = \frac{K_{IC}^2}{E_0} + p_c\delta_c \quad (17)$$

it follows from Equations 8, 16 and 17 that with

$\lambda = 8p_c^2 a_0/\pi K_{IC}^2$

$$\frac{J}{G_F} = \frac{1 + \lambda\{2 + \ln[\lambda + (\lambda^2 + 1)^{1/2}]\}}{G_F/J_{IC}} \quad (18)$$

where $J_{IC} = K_{IC}^2/E_0$. G_F is the specific fracture energy for the softening law used in this paper, while J_{IC} represents the contribution due to this energy arising from the pre-tail region of the softening law. Equation 18 is valid provided that the stress maximum is attained prior to the full development of a softening zone, i.e. provided that (see Inequality 10)

$$[1 + \lambda\{2 + \ln(\lambda + (\lambda^2 + 1)^{1/2})\}] < \frac{G_F}{J_{IC}} \quad (19)$$

For values of λ which do not satisfy this inequality, i.e. when the stress maximum is associated with the full development of a softening zone, J/G_F is equal to unity. The ratio J/G_F is plotted in Fig. 5 as a function of λ for specific values of the parameter $\beta = G_F/J_{IC}$.

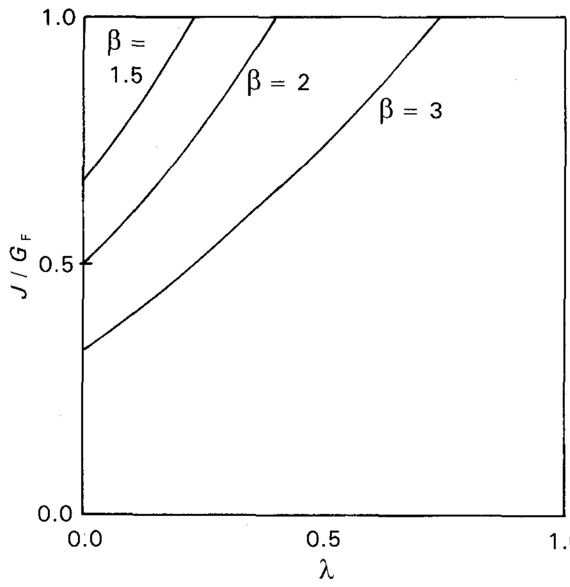

Figure 5 The value of the J-integral at the attainment of maximum load. $\lambda = 8p_c^2 a_0/\pi K_{IC}^2 = 8p_c^2 a_0/\pi E_0 J_{IC}$ and $\beta = G_F/J_{IC} = K_{m\infty}^2/K_{IC}^2$.

3. Discussion

The preceding analysis was concerned with the criterion for the attainment of maximum load for a cracked solid where the material is of the elastic softening variety. A simple description was used to describe the material's softening behaviour, there being a non-zero K_{IC} at the crack tip coupled with a constant stress within the softening zone. It has been clearly shown that the maximum load can be attained prior to the full development of a softening zone, provided.that the initial crack size is less than a critical value that is dependent upon the material's softening characteristics. A consequence is that the value of the J-integral at the attainment of maximum load is not invariant; for a specific softening law its value is dependent on crack size (see Fig. 5).

It should be emphasized that this paper's conclusion regarding the effect of initial crack size on the potential for the attainment of maximum load prior to the full development of a softening zone has been reached by employing a very simple description for the material's softening behaviour. It has been assumed that K_{IC} is non-zero at the crack tip and that there is a constant stress within the softening zone. However, the conclusion should also be applicable to a material for which there is a rapid fall-off from a high stress value to a much lower value at a small displacement, followed by a long tail in the softening law. This comment is made with the knowledge that Stahle [10] has shown that the attainment of maximum load can occur before the full development of a softening zone with a linear softening force law with a maximum stress p_c. Stahle also showed that as the initial crack size decreases, there is a greater tendency for the maximum load to be attained prior to the full development of a softening zone; this result is in accord with the present paper's findings.

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